# OPERATOR INEQUALITIES RELATED TO WEAK 2-POSITIVITY

MOHAMMAD SAL MOSLEHIAN[1] AND JUN ICHI FUJII[2]

ABSTRACT. In this paper we introduce the notion of weak 2-positivity and present some examples. We establish some operator Cauchy–Schwarz inequalities involving the geometric mean and give some applications. In particular, we present some operator versions of Hua's inequality by using the Choi–Davis–Jensen inequality.

## 1. INTRODUCTION

Let $\mathbb{B}(\mathscr{H}), \langle \cdot, \cdot \rangle)$ stand for the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ and let $I$ denote the identity operator. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the full matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive (positive-semidefinite for matrices) if $\langle A\xi, \xi \rangle \geq 0$ holds for every $\xi \in \mathscr{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. A map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is said to be *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$. A map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is called 2-*positive* if the map $\Phi_2 : \mathbb{B}(\mathscr{H} \oplus \mathscr{H}) \to \mathbb{B}(\mathscr{K} \oplus \mathscr{K})$ defined by $\Phi_2([A_{ij}]_{2\times 2}) = [\Phi(A_{ij})]_{2\times 2}$ takes each positive block matrix to a positive one. If $\Phi_2$ preserves the positivity of each block matrix of the form $\begin{bmatrix} A & C \\ C & B \end{bmatrix}$, then we call $\Phi$ *weakly* 2-*positive*. We say that $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is a $*$-map if $\Phi(A^*) = (\Phi(A))^*$. Choi [4, Corollary 4.4] showed that a positive linear map is weakly 2-positive. On the other hand, the Moore–Penrose inverse $^\dagger$ on the matrix algebra $\mathcal{M}_n$ gives a map $\Phi^\dagger$ defined by $\Phi^\dagger(A) = A^\dagger$, which is a nonlinear positive map while it is not weakly 2-positive (and so not 2-positive). In fact, since $\Phi^\dagger$ assigns the inverses to invertible matrices, we have

$$\begin{bmatrix} 2I & I \\ I & 2I \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \otimes I \geq 0 \quad \text{while} \quad \begin{bmatrix} \Phi^\dagger(2I) & \Phi^\dagger(I) \\ \Phi^\dagger(I) & \Phi^\dagger(2I) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \otimes I \ngeq 0.$$

Next we present a non-trivial example of a weakly 2-positive map, which is not 2-positive. Let us recall a useful criterion due to Ando [1, Theorem I.1]. It states that a block matrix $T =$

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$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ is positive if and only if there exists a contraction $W$ such that $C = A^{\frac{1}{2}}WB^{\frac{1}{2}}$. We first note that the nonlinear map $X \mapsto (\det X)I$ on $\mathcal{M}_n$ is 2-positive. In fact, the condition $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \geq 0$ implies that $C = A^{\frac{1}{2}}WB^{\frac{1}{2}}$ for some contraction $W$. Then $|\det W| \leq 1$ and $\det C = \sqrt{\det A}\det W\sqrt{\det B}$. Using again the above criterion we conclude that $\Phi$ is 2-positive. The map $\Phi_\alpha(X) = X^* + \alpha(\det X)I$ for $\alpha \geqq 0$ is neither linear nor conjugate linear. It is clearly weakly 2-positive. Moreover, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $A^{\frac{1}{2}} = A$, $B^{\frac{1}{2}} = B/2$ and $C = A^{\frac{1}{2}}IB^{\frac{1}{2}}$, so that $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \geq 0$. Noting to $\det A = \det B = \det C = 0$, we have

$$\begin{pmatrix} \Phi_\alpha(A) & \Phi_\alpha(C) \\ \Phi_\alpha(C^*) & \Phi_\alpha(B) \end{pmatrix} = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} + \alpha \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

which is not positive since its determinant is negative. Therefore $\Phi_\alpha$ is not 2-positive for any $\alpha \geqq 0$. Furthermore, these matrices $A, B, C$ can be used to show that the transpose map $\Phi(A) = A^{tr}$ on $\mathcal{M}_2$ is a weakly 2-positive linear map that is not 2-positive.

The geometric mean $A\#B$ of two positive operators $A, B \in \mathbb{B}(\mathcal{H})$ is characterized by Ando [1]

$$A\#B = \max\left\{ X = X^* \in \mathbb{B}(\mathcal{H}) : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \right\}.$$

Then we immediately have $\Phi(A\#B) \leq \Phi(A)\#\Phi(B)$ for any weakly 2-positive map $\Phi$. Ando [1] also characterized the harmonic mean $A\,!\,B$ by

$$A\,!\,B = \max\left\{ X = X^* \in \mathbb{B}(\mathcal{H}) : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix} \right\}.$$

Then, for a weakly 2-positive map $\Phi$, we have

$$\begin{bmatrix} \Phi(2A - A\,!\,B) & \Phi(-A\,!\,B) \\ \Phi(-A\,!\,B) & \Phi(2B - A\,!\,B) \end{bmatrix} \geq 0.$$

If $\Phi$ is linear in this case, we have

$$\begin{bmatrix} 2\Phi(A) & 0 \\ 0 & 2\Phi(B) \end{bmatrix} \geq \begin{bmatrix} \Phi(A\,!\,B) & \Phi(A\,!\,B) \\ \Phi(A\,!\,B) & \Phi(A\,!\,B) \end{bmatrix},$$

and hence $\Phi(A\,!\,B) \leq \Phi(A)\,!\,\Phi(B)$ holds, which is shown in [1, Cor.IV.1.3].

In this note we present operator Cauchy–Schwarz inequalities for 2-weakly positive and 2-positive maps involving the operator geometric mean and give two operator Hua types inequalities as application.

## 2. Cauchy–Schwarz type inequalities

One of the fundamental inequalities in mathematics is the Cauchy–Schwarz inequality. It states that in an inner product space $(\mathscr{X}, \langle \cdot, \cdot \rangle)$

$$|\langle x, y \rangle| \leq \|x\| \|y\| \qquad (x, y \in \mathscr{X}).$$

There are many generalizations and applications of this inequality for integrals and isotone functionals; see the monograph [6]. Moreover, some Cauchy–Schwarz inequalities for Hilbert space operators and matrices involving unitarily invariant norms were given by Jocić [14] and Kittaneh [16]. Also Joiţa [15], Ilišević and Varošanec [13], the first author and Persson [18], Arambašić, Bakić and the first author [2] have investigated the Cauchy–Schwarz inequality and its various reverses in the framework of $C^*$-algebras and Hilbert $C^*$-modules. Tanahashi, A. Uchiyama and M. Uchiyama [20] investigated some Schwarz type inequalities and their converses in connection with semi-operator monotone functions. A refinement of the Cauchy–Schwarz inequality involving connections is investigated by Wada [21]. An application of the covariance-variance inequality to the Cauchy–Schwarz inequality was obtained by Fujii, Izumino, Nakamoto and Seo [10]. Some operator versions of the Cauchy–Schwarz inequality with simple conditions for the case of equality are presented by the second author [9].

To achieve our main result we need the polar decomposition of bounded linear operators. Recall that if $A \in \mathbb{B}(\mathscr{H})$, then there exists a unique partial isometry $U \in \mathbb{B}(\mathscr{H})$ such that $A = U|A|$ and $\ker(U) = \ker(|A|)$ (the polar decomposition). Then $U^*A = |A|$ and $A^* = U^*|A^*|$ is the polar decomposition of $A^*$.

**Theorem 2.1.** *Let $A, B, X, Y \in \mathbb{B}(\mathscr{H})$ be arbitrary operators.*

  (i) *If $\Phi$ is a weakly 2-positive map, then $\Phi(|X^*A^*Y|) \leq \Phi(V^*X^*|A|XV) \,\#\, \Phi(Y^*|A^*|Y)$, in which $X^*A^*Y = V|X^*A^*Y|$ denotes the polar decomposition.*

  (ii) *If $\Phi$ is a 2-positive $*$-map, then $|\Phi(X^*A^*Y)| \leq U^*\Phi(X^*|A|X)U \,\#\, \Phi(Y^*|A^*|Y)$, in which $\Phi(X^*A^*Y) = U|\Phi(X^*A^*Y)|$ denotes the polar decomposition.*

*Proof.* (i) First note that

$$
\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} |A|^{1/2} & 0 \\ |A|^{1/2} & 0 \end{bmatrix} \begin{bmatrix} |A|^{1/2} & |A|^{1/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}^* \geq 0,
$$

where we apply the polar decomposition $A = W|A|$. Hence

$$
\begin{bmatrix} X^*|A|X & X^*A^*Y \\ Y^*AX & Y^*|A^*|Y \end{bmatrix} = \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix} \begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq 0. \tag{2.1}
$$

Utilizing the polar decomposition $X^*A^*Y = V|X^*A^*Y|$ we obtain

$$
\begin{aligned}
\begin{bmatrix} V^*(X^*|A|X)V & |X^*A^*Y| \\ |X^*A^*Y| & Y^*|A^*|Y \end{bmatrix} &= \begin{bmatrix} V^*(X^*|A|X)V & V^*(X^*A^*Y) \\ (Y^*AX)V & Y^*|A^*|Y \end{bmatrix} \\
&= \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X^*|A|X & X^*A^*Y \\ Y^*AX & Y^*|A^*|Y \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} \\
&\geq 0.
\end{aligned}
$$

Due to the weak 2-positivity of $\Phi$, we get

$$
\begin{bmatrix} \Phi(V^*(X^*|A|X)V) & \Phi(|X^*A^*Y|) \\ \Phi(|X^*A^*Y|) & \Phi(Y^*|A^*|Y) \end{bmatrix} \geq 0.
$$

Thus we obtain

$$
\Phi(|X^*A^*Y|) \leq \Phi(V^*X^*|A|XV) \,\#\, \Phi(Y^*|A^*|Y).
$$

(ii) It follows from (2.1) and 2-positivity of $\Phi$ that

$$
\begin{bmatrix} \Phi(X^*|A|X) & \Phi(X^*A^*Y) \\ \Phi(Y^*AX) & \Phi(Y^*|A^*|Y) \end{bmatrix} \geq 0,
$$

whence, by using the polar decoposition $\Phi(X^*A^*Y) = U|\Phi(X^*A^*Y)|$, we get

$$
\begin{aligned}
\begin{bmatrix} U^*\Phi(X^*|A|X)U & |\Phi(X^*A^*Y)| \\ |\Phi(X^*A^*Y)| & \Phi(Y^*|A^*|Y) \end{bmatrix} &= \begin{bmatrix} U^*\Phi(X^*|A|X)U & U^*\Phi(X^*A^*Y) \\ \Phi(Y^*AX)U & \Phi(Y^*|A^*|Y) \end{bmatrix} \\
&= \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \Phi(X^*|A|X) & \Phi(X^*A^*Y) \\ \Phi(Y^*AX) & \Phi(Y^*|A^*|Y) \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \\
&\geq 0,
\end{aligned}
$$

which gives the desired inequality. $\qquad\square$

*Remark* 2.2. The proof of Theorem 2.1(ii) shows that if $A = A^*$ and $Y = X$, then the "2-positivity" of $\Phi$ can be replaced by the weaker assumption "weak 2-positivity". Then we get (ii)$'$ If $\Phi$ is a weakly 2-positive $*$-map, then $|\Phi(X^*AX)| \leq U^*\Phi(X^*|A|X)U \# \Phi(X^*|A|X)$, in which $\Phi(X^*AX) = U|\Phi(X^*AX)|$ denotes the polar decomposition.

Now consider the separable Hilbert space $H = \ell_2$. Take the 2-positive map $\Phi(A) = \langle Ae, e \rangle$ where $A \in \mathbb{B}(\mathscr{H})$, $e = (1, 0, 0, \cdots)$ and $X = x \otimes \overline{e}$, $Y = y \otimes \overline{e}$ where $(x \otimes \overline{y})(z) := \langle z, y \rangle x$. Then we get from Theorem 2.1 (ii) the following Cauchy–Schwarz inequality in Hilbert spaces:

**Corollary 2.3.** *Let $A \in \mathbb{B}(\mathscr{H})$ and $x, y \in \mathscr{H}$. Then*

$$|\langle Ax, y \rangle|^2 \leq \langle |A|x, x \rangle \langle |A^*|y, y \rangle.$$

Considering the positive linear functional $\mathrm{tr}(\cdot)$ on $\mathcal{M}_n$, it follows from Theorem 2.1(i) that

**Corollary 2.4.** *Let $A, X, Y \in \mathcal{M}_n$. Then*

$$\mathrm{tr}(|X^*A^*Y|)^2 \leq \mathrm{tr}(X^*|A|X)\mathrm{tr}(Y^*|A^*|Y).$$

**Corollary 2.5.** *Let $X \in \mathbb{B}(\mathscr{H})$.*

(i) *If $\Phi$ is a weakly 2-positive map, then $\Phi(|X|) \leq \Phi(V^*|X^*|V) \# \Phi(|X|)$, where $X = V|X|$ is the polar decomposition.*

(ii) *If $\Phi$ is a 2-positive $*$-map, then $|\Phi(X)| \leq U^*\Phi(|X^*|^{1/2})U \# \Phi(|X|^{3/2})$, where $\Phi(X) = U|\Phi(X)|$ is the polar decomposition.*

*Proof.* Let $X = V|X|$ be the polar decomposition of $X$. It follows from Theorem 2.1 (i) that

(i) $\Phi(|X|) = \Phi(|V^*XI|) \leq \Phi(IV^*|X^*|VI) \# \Phi(|X|) = \Phi(V^*|X^*|V) \# \Phi(|X|)$.

(ii) Utilizing Theorem 2.1 (ii) we have

$$\begin{aligned} |\Phi(X)| &= |\Phi(V|X|^{1/2}|X|^{1/2})| \\ &\leq U^*\Phi(V|X|^{1/2}V^*)U \# \Phi(|X|^{1/2}|X|^{1/2}|X|^{1/2}) \\ &= U^*\Phi(|X^*|^{1/2})U \# \Phi(|X|^{3/2}). \end{aligned}$$ $\qquad\square$

## 3. Applications to Hua's inequality

Hua's inequality states that

$$\left(\delta - \sum_{i=1}^{n} x_i\right)^2 + \alpha \sum_{i=1}^{n} x_i^2 \geq \frac{\alpha}{n+\alpha}\delta^2,$$

where $\delta$, $\alpha$ are positive numbers and $x_i$ $(i = 1, 2, \ldots, n)$ are real numbers. There are several refinement and improvement of this inequality in the literature; see [17] and references

therein. An operator version of Hua's inequality was given by Drnovšek [7]. Moreover, Radas and Šikić [19] generalized the Hua inequality for linear operators in real inner product spaces. A refinement of Hua's inequality was presented by the second author in [8] by showing that if $A, B$ are bounded linear operators acting on a Hilbert space $\mathscr{H}$ and $\varphi$ is a state on $\mathbb{B}(\mathscr{H})$, then

$$(1 - |\varphi(B^*A)|)^2 \geq (1 - \sqrt{\varphi(A^*A)\varphi(B^*B)})^2 \geq \varphi(I - A^*A)\varphi(I - B^*B), \qquad (3.1)$$

which in turn gives an extension of the above classical Hua's inequality by considering $\varphi$ as the normalized trace on the matrix algebra $\mathcal{M}_n$ and some suitable diagonal matrices. An extension in the setting of Hilbert $C^*$-modules and operators on Hilbert spaces was given by the first author in [17].

Our first result in this section gives an extension of (3.1). Recall that a contraction is an operator $A$ of norm less than or equal one.

**Theorem 3.1.** *Let $\Phi$ be a 2-positive $*$-map and let $A, B, X, Y \in \mathbb{B}(\mathscr{H})$ be arbitrary operators. If $\Phi(X^*A^*Y) = U|\Phi(X^*A^*Y)|$ is the polar decomposition of $\Phi(X^*A^*Y)$, and $\Phi(Y^*|A^*|Y)$ and $\Phi(X^*|A|X)$ are contractions, then*

$$I - |\Phi(X^*A^*Y)| \geq U^*\big(I - \Phi(X^*|A|X)\big)U \,\#\, \big(I - \Phi(Y^*|A^*|Y)\big).$$

*Proof.* Theorem 2.1 (ii) ensures that

$$I - |\Phi(X^*A^*Y)| \geq I - \big(U^*\Phi(X^*|A|X)U \,\#\, \Phi(Y^*|A^*|Y)\big). \qquad (3.2)$$

Using the properties of the geometric mean (see [11, Chapter 5]), we get

$$\big(U^*(I - \Phi(X^*|A|X))U \,\#\, \big(I - \Phi(Y^*|A^*|Y)\big)\big) + \big(U^*\Phi(X^*|A|X)U \,\#\, \Phi(Y^*|A^*|Y)\big)$$

$$\begin{aligned} &\leq\ U^*U\#I &&\text{(by the subadditivity of the geometric mean)}\\ &\leq\ I\#I &&\text{(by the monotonicity of the geometric mean)}\\ &=\ I\,, \end{aligned}$$

which together with (3.2) give the required inequality. $\qquad\square$

Now let $f$ be a continuous real function $f$ defined on an interval $\mathcal{J} \subseteq \mathbb{R}$. The function $f$ is called *operator convex* if

$$f\left(\frac{A+B}{2}\right) \leq \frac{f(A) + f(B)}{2}$$

for all selfadjoint operators $A$ and $B$ with spectra contained in $\mathcal{J}$. There are several statements equivalent to the operator convexity; see [11, Theorems 1.9 and 1.10]. In particular,

$f$ is operator convex if and only if

$$f\left(\sum_{i=1}^{n} X_i^* A_i X_i\right) \leq \sum_{i=1}^{n} X_i^* f(A_i) X_i \qquad (3.3)$$

for all self-adjoint bounded operators $A_i$ with spectra contained in $\mathcal{J}$ and all bounded operators $X_i$ with $\sum_{i=1}^{n} X_i^* X_i = I$; cf [12]. The Jensen operator inequality due to Davis [5] and Choi [3] reads as follows

$$f(\Phi(A)) \leq \Phi(f(A)) \qquad \text{(The Choi–Davis–Jensen inequality)}$$

where $\Phi$ is a unital positive linear map on $\mathbb{B}(\mathscr{H})$, $f$ is operator convex and $A$ is a self-adjoint operator whose spectrum $\mathrm{sp}(A)$ is contained in $\mathcal{J}$.

Finally we show another type of Hua's operator inequality. Recall that a *conditional expectation* $\Phi$ from a unital $C^*$-algebra $\mathscr{A}$ of operators to a $C^*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$ containing its identity is a linear norm reducing idempotent. Such a map is completely positive and satisfies the bimodule property $\Phi(AXB) = A\Phi(X)B$ for all $A, B \in \mathscr{B}$ and $X \in \mathscr{A}$.

**Theorem 3.2.** *Let $f$ be an operator convex function on an interval $\mathcal{J}$ and $\Phi$ be a conditional expectation from a unital $C^*$-algebra $\mathscr{A}$ of operators to a $C^*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$ containing its identity. If $C \in \mathscr{B}$ is invertible and $B \in \mathscr{A}$ is self-adjoint and satisfies*

$$\mathrm{sp}(I - \Phi(B)) \cup \mathrm{sp}((I + C^*C)^{-1}) \cup \mathrm{sp}(C^{*-1}BC^{-1}) \subseteq \mathcal{J},$$

*then*

$$f(I - \Phi(B)) + C^*\Phi\left(f\left(C^{*-1}BC^{-1}\right)\right)C \geq f\left((I + C^*C)^{-1}\right)(I + C^*C).$$

*Proof.* Put

$$X = (I + C^*C)^{-1/2} \quad and \quad Y = C(I + C^*C)^{-1/2}.$$

Then $X^*X + Y^*Y = I$. We have

$$
\begin{aligned}
f(I &- \Phi(B)) + C^*\Phi(f(C^{*-1}BC^{-1}))C \\
&= X^{-1}\left[Xf(I - \Phi(B))X + Y^*\Phi(f(C^{*-1}BC^{-1}))Y\right]X^{-1} \\
&\geq X^{-1}\left[Xf(I - \Phi(B))X + Y^*f(\Phi(C^{*-1}BC^{-1}))Y\right]X^{-1} \\
&\qquad\qquad\qquad\qquad (\text{ by the Choi–Davis–Jensen inequality}) \\
&= X^{-1}\left[Xf(I - \Phi(B))X + Y^*f(C^{*-1}\Phi(B)C^{-1})Y\right]X^{-1} \\
&\qquad\qquad\qquad\qquad (\text{by the bimodule property of } \Phi) \\
&\geq X^{-1}f\left(X(I - \Phi(B))X + Y^*C^{*-1}\Phi(B)C^{-1}Y\right)X^{-1} \\
&\qquad\qquad\qquad\qquad (\text{by } X^*X + Y^*Y = I \text{ and } (3.3)) \\
&= X^{-1}f\left(X(I - \Phi(B))X + X\Phi(B)X\right)X^{-1} \\
&= f(X^2)X^{-2} \qquad\qquad\qquad (\text{by the functional calculus}) \\
&= f\left((I + C^*C)^{-1}\right)(I + C^*C).
\end{aligned}
$$

$\square$

**Corollary 3.3.** *Let $f$ be an operator convex function on an interval $\mathcal{J}$, $\varphi$ be a state and $\gamma > 0$. If $B$ is self-adjoint, $1 - \varphi(B)$ and $1/(\gamma + 1)$ belong to $\mathcal{J}$ and $\mathrm{sp}(B/\gamma) \subseteq \mathcal{J}$, then*

$$
f(1 - \varphi(B)) + \gamma\varphi\big(f(B/\gamma)\big) \geq (1 + \gamma)f\left(\frac{1}{1 + \gamma}\right).
$$

## References

[1] T. Ando, *Topics on operator inequality*, Hokkaido Univ. Lecture Note, 1978.

[2] Lj. Arambasić, D. Bakić and M.S. Moslehian, *A treatment of the Cauchy–Schwarz inequality in $C^*$-modules*, J. Math. Anal. Appl. **381** (2011) 546–556.

[3] M.-D. Choi, *A Schwartz inequality for positive linear maps on $C^*$-algebras*, Illinois J. Math. **18** (1974), 565–574.

[4] M.-D. Choi, *Some assorted inequalities for positive linear maps on $C^*$-algebras*, J Operator Theory **4** (1980), 271–285.

[5] C. Davis, *A Schwartz inequality for convex operator functions*, Proc. Amer. Math. Soc. **8** (1957), 42–44.

[6] S.S. Dragomir, *A survey on Cauchy–Bunyakovsky–Schwarz type discrete inequalities*, J. Inequal. Pure Appl. Math., **4** (2003), no. 3, Article 63, 142 pp.

[7] R. Drnovšek, *An operator generalization of the Lo-Keng Hua inequality*, J. Math. Anal. Appl., **196** (1995), no.3, 1135–1138.

[8] J.I. Fujii, *Operator inequalities for Schwarz and Hua*, Sci. Math. **2** (1999), no. 3, 263–268.

[9] J.I. Fujii, *Operator-valued inner product and operator inequalities*, Banach J. Math. Anal. **2** (2008), no. 2, 59–67.

[10] M. Fujii, S. Izumino, R. Nakamoto and Y. Seo, *Operator inequalities related to Cauchy-Schwarz and Holder-McCarthy inequalities*, Nihonkai Math. J. **8** (1997) 117-122.

[11] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond–Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.

[12] F. Hansen and G.K. Pedersen, *Jensen's operator inequality*, Bull. London Math. Soc. **35** (2003), no. 4, 553–564.

[13] D. Ilišević and S. Varošanec, *On the Cauchy–Schwarz inequality and its reverse in semi-inner product $C^*$-modules*, Banach J. Math. Anal. **1** (2007), 78–84.

[14] D.R. Jocić, *Cauchy–Schwarz and means inequalities for elementary operators into norm ideals*, Proc. Amer. Math. Soc. **126** (1998), no. 9, 2705-2711.

[15] M. Joiţa, *On the Cauchy–Schwarz inequality in $C^*$-algebras*, Math. Rep. (Bucur.) **3(53)** (2001), no. 3, 243–246.

[16] F. Kittaneh, *Some norm inequalities for operators*, Canad. Math. Bull. **42** (1999), no. 1, 87-96.

[17] M.S. Moslehian, *Operator extensions of Hua's inequality*, Linear Algebra Appl. **430** (2009), no. 4, 1131–1139.

[18] M.S. Moslehian and L.-E. Persson, *Reverse Cauchy–Schwarz inequalities for positive $C^*$-valued sesquilinear forms*, Math. Inequal. Appl. **12** (2009), no. 4, 701–709.

[19] S. Radas and T. Šikić, *A note on the generalization of Hua's inequality*, Tamkang J. Math., **28** (1997), no.4, 321–323.

[20] K. Tanahashi, A. Uchiyama and M. Uchiyama, *On Schwarz type inequalities*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2549–2552.

[21] S. Wada, *On some refinement of the Cauchy–Schwarz inequality*, Linear Algebra Appl. **420** (2007), no. 2-3, 433-440

[1] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.

*E-mail address*: moslehian@ferdowsi.um.ac.ir and moslehian@member.ams.org

[2] Department of Art and Sciences (Information Science), Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.

*E-mail address*: fujii@cc.osaka-kyoiku.ac.jp